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Candidates for anti-de Sitter-horizons

Koushik Ray*

Dipartimento di Fisica
Università di Roma "Tor Vergata"
INFN — Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00173 Rome ITALY
E-mail: koushik@roma2.infn.it

Abstract: We find, from the toric description of the moduli space of D3–branes on non-compact six-dimensional singularities $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$ in the blown-down limit, the four-dimensional bases on which these singular spaces are complex cones, and prove the existence of Kähler–Einstein metrics on these four-dimensional bases. This shows, in particular, that one can use the horizons obtained from these base spaces by a U(1)-foliation as compact parts of the target space for Type–IIB string theory with AdS^5 in the context of the AdS-CFT correspondence.

Keywords: p-branes, D-branes, Brane Dynamics in Gauge Theories.

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*INFN–Fellow.

Contents

1. Introduction

The conjecture [1, 2] on the nexus between Type–IIB superstring theory on $AdS^5\times\mathcal{H}$, where AdS^5 is the five-dimensional anti-de Sitter space and $\mathcal{H}$ is a five-dimensional variety, called *horizon* in the sequel, and superconformal field theories on the four-dimensional boundary of the five-dimensional anti-de Sitter space, thought of as a theory of D3–branes, has instigated a new spate of research. The conjecture, known by now as the AdS-CFT correspondence, has already been tested in sundry instances. The tests can be broadly classified into two categories. The first is a consideration of symmetries of the two candidates. Apart from a diagonal U(1) subgroup of the gauge group, corresponding to a free-photon in the gauge theory [1, 3, 4], the group of global isometries of the supergravity background in the presence of suitable four-form fluxes corresponding to the Type–IIB superstring theory is identified with the R-symmetry group of the deemed superconformal theory on the four-dimensional boundary of AdS^5 (see, for example, [5]). The other consists in an identification of operators in the two theories and in the comparison of their correlation functions (see, for example, [6, 3, 7]). When the five-dimensional horizon $\mathcal{H}$ is the five-sphere, $\mathbb{S}^5$, the conformal counterpart on the D3–brane is the $\mathcal{N}=4$ supersymmetric gauge theory of the boundary of AdS^5. The horizon $\mathcal{H}$ is envisaged as being part of the space, namely $\mathbb{C}^3$, transverse to the world-volume of the D3–brane.

One of the interesting courses of development in the subject is a generalisation of the original conjecture [1] to models with less than maximal supersymmetry [2, 8, 9, 4]. Considering a theory of D3–branes on a Gorenstein canonical singularity of, for example, the type $\mathbb{C}^3/\Gamma$, where Γ is a discrete subgroup of SU(3), it is possible to break some of the supersymmetries of the field theory of the brane. The corresponding candidate dual theory is then the Type–IIB theory compactified on $AdS^5 \times \mathbb{S}^5/\Gamma$ [8, 2]. For example, if Γ is isomorphic to a $\mathbb{Z}_2$ subgroup of an SU(2) subgroup of SU(3), then the resulting dual theory has been found to be a four-dimensional $\mathcal{N} = 2$ gauge theory [11, 9, 10].

Theories of a D3–brane on orbifolds of $\mathbb{C}^3$ of the form $\mathbb{C}^3/\Gamma$ have been considered earlier, for different groups Γ. Examples include discrete groups Γ isomorphic to $\mathbb{Z}_k$, for $k = 3, 5$ [12], 7, 9, 11 [13], to $\mathbb{Z}_2 \times \mathbb{Z}_2$ without discrete torsion [14, 15] and with discrete torsion [16, 17], as well as some cases where Γ is non-abelian [18, 19]. In some of these analyses [12, 13, 14, 15, 18, 19] the moduli space of a D3–brane at an orbifold singularity, derived as a solution to the F- and D-flatness conditions of the corresponding supersymmetric gauge theory of the D3–brane, admits a toric geometric description. It is interesting to enquire whether one can also use these other cases to derive an admissible horizon for AdS^5.

For an appropriate action of a U(1) group on the horizon $\mathcal{H}$, compatible with the complex structure of its metric cone $C(\mathcal{H})$, one can obtain the toric data of the base $\mathcal{H}_B = \mathcal{H}/\mathrm{U}(1)$ from the toric data of the moduli space of the D3–brane on $\mathbb{C}^3/\Gamma$ [4]. It has been shown [20] that if the space $\mathcal{H}_B$ admits a Kähler-Einstein metric, then one can, from the knowledge of the U(1)-action, find the horizon $\mathcal{H}$ by considering U(1)-bundles of $\mathcal{H}_B$. This has been illustrated for several examples arising from partial resolutions of a $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity [4].

In order to obtain the horizon from the moduli space of a D3–brane, one starts from the field theory on the D3–brane on $\mathbb{C}^3/\Gamma$. This field theory is the blown-down limit of the theory of a D3–brane on the Calabi–Yau singularity. In order for this theory to be dual to the Type–IIB theory on $AdS^5 \times \mathcal{H}$, the horizon $\mathcal{H}$ is taken to be a space on which $\mathbb{C}^3/\Gamma$ is a cone. It has been shown that in order to retain some supersymmetry in the quotient theory, the horizon has to be a Sasaki–Einstein manifold [21, 20] and correspondingly, the manifold $\mathcal{H}_B = \mathcal{H}/\mathrm{U}(1)$ must be Kähler–Einstein. If the group-action of U(1) on the horizon $\mathcal{H}$ is *regular*, meaning that the orbits of the U(1)-action are closed and have the same length, the base $\mathcal{H}_B$ of this U(1)-foliation admits a Kähler–Einstein metric [22, 4]. See [23] for results with a weaker condition of a quasi-regular U(1)-action. If the discrete group Γ is isomorphic to $\mathbb{Z}_3$, for example, then the group action of U(1) is regular on the horizon [22, 4]. However, if the U(1)-action on the horizon is not regular, the issue of the existence of Kähler–Einstein metrics on the corresponding bases needs be settled on a case by case basis.

In this note we consider the question of the existence of Kähler–Einstein metrics on the base $\mathcal{H}_B$ of the D3–brane moduli spaces in two examples, namely D3–branes on $\mathbb{C}^3/\Gamma$, where Γ is a discrete subgroup of SU(3), isomorphic to $\mathbb{Z}_3$ and $\mathbb{Z}_5$, that is for the orbifolds $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$. The first corresponds to a horizon that admits a regular U(1)-action, while the other does not. The corresponding theories of D3-branes have been studied earlier [12]. In both cases we work out the toric data for the base, starting from the toric data of the orbifolds, obtained earlier [12]. We then prove the existence of a Kähler–Einstein metric on the respective four-dimensional bases. For the base of $\mathbb{C}^3/\mathbb{Z}_3$, the canonical toric metric is the Fubini-Study metric on $\mathbb{CP}^2$, which is Kähler–Einstein. The canonical toric metric on the base of $\mathbb{C}^3/\mathbb{Z}_5$ is, however, Kähler, but not Einstein. We find out the condition for the existence of a Kähler–Einstein metric on $\mathcal{H}_B$ in terms of a deformation of the canonical Kähler potential, following [25, 26]. This leads to a linear partial differential equation whose solution purveys a Kähler–Einstein metric on the corresponding base manifold $\mathcal{H}_B$, thereby establishing the existence of a Kähler–Einstein metric on $\mathcal{H}_B$.

The structure of this note is as follows. In section 2, we briefly review the argument establishing the necessity of a Kähler–Einstein metric on $\mathcal{H}_B$, following [20]. In section 3, we review the necessary information and formulas for the calculations, following [24, 25, 26]. Finally in sections 4 and 5, we present the calculation of Kähler–Einstein metrics on the bases of $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$, before concluding in section 6.

2. The cone, the horizon & the base

Given a space $\mathcal{H}$, the *metric cone* $C(\mathcal{H})$ on $\mathcal{H}$ is defined by the warped product of $\mathcal{H}$ and the half-line, $\mathbb{R}^+$, as $C(\mathcal{H}) = \mathcal{H}\times_{r^2}\mathbb{R}^+$, endowed with the metric $\langle\cdots\rangle_{C(\mathcal{H})} = dr^2 + r^2\langle\cdots\rangle_{\mathcal{H}}$. Here and below we use subscripts to metrics descriptively whenever convenient. The conical nature of $C(\mathcal{H})$ is discerned by noting that there exists a group of diffeomorphisms of $C(\mathcal{H})$, isomorphic to the multiplicative group of real numbers, $\mathbb{R}_+$, namely, $r \longmapsto\ t\, r$, for any positive real number t, which rescales the metric as: $\langle\cdots\rangle_{C(\mathcal{H})} \longmapsto t^2\langle\cdots\rangle_{C(\mathcal{H})}$. Moreover, if the space $\mathcal{H}$ admits a transitive U(1)-action, then one can define another space $\mathcal{H}_B$ by quotienting $\mathcal{H}$ by the U(1). Thus, the space $\mathcal{H}_B$ can be written as $C(\mathcal{H})/(\mathbb{R}_+\times \mathrm{U}(1))$, that, in view of the decomposition $\mathbb{C}^\star = \mathbb{R}_+\times\mathrm{U}(1)$, can be expressed as $C(\mathcal{H})/\mathbb{C}^\star$. Here $\mathbb{C}^\star$ denotes the complex-plane with the origin deleted, that is, $\mathbb{C}^\star = \mathbb{C}\setminus\{0\}$, or, in other words, the algebraic torus. The metric cone on $\mathcal{H}$ can therefore be viewed as a *complex cone* over $\mathcal{H}_B$. In the sequel we shall refer to the metric cone simply as a *cone*, while the complex cone will be explicitly mentioned. Also, we shall refer to the space $\mathcal{H}$ as the *horizon* whilst $\mathcal{H}_B$ will be referred to as the *base* (of the complex cone as well as of the horizon).

Let us begin by briefly recalling some features of supergravity background solutions in the presence of D3–branes preserving some supersymmetry. The solution to the supergravity equations of motion for the ten-dimensional metric retaining some supersymmetry is sought in the following form [20]

$$ds^2 = e^{2\Upsilon(x^m)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2\Upsilon(x^m)}h_{mn}dx^m dx^n\,, \qquad \begin{array}{r} m,n = 1,\ldots,6\,, \\ \mu,\nu = 7,\ldots,10\,, \end{array} \tag{2.1}$$

where the ten-dimensional space-time is taken to be split into two parts. Four of the ten directions are parallel to the world-volume of the D3–brane, coordinatized by x^μ, $\mu = 7, \ldots, 10$, and $\eta_{\mu\nu}$ denotes the flat-metric. The remaining six coordinates are transverse to the D3–brane are denoted by x^m, $m = 1, \ldots, 6$. Thus, the D3–brane is a point in the six-dimensional space coordinatized by x^m, $m = 1, \ldots, 6$. In (2.1), $\Upsilon(x^m)$ is a harmonic function of the coordinates of the six-dimensional space transverse to the world-volume of the D3–brane. The six-dimensional metric is chosen to be of the form

$$ds^2_{C(\mathcal{H})} = h_{mn} dx^m dx^n = dr^2 + r^2 g_{ij} dx^i dx^j \,, \qquad i, j = 1, \ldots, 5 \,, \tag{2.2}$$

where the five-dimensional metric g_{ij} is the metric on the horizon $\mathcal{H}$ that appears in the compactification of Type–IIB supergravity on $AdS^5 \times \mathcal{H}$. Finally, the horizon is envisaged as a U(1)-bundle over a four-dimensional space $\mathcal{H}_B$, with metric of the form

$$ds^2_{\mathcal{H}} = g_{ij} dx^i dx^j = ds^2_{\mathcal{H}_B} + 4(d\psi + A_a dx^a)^2 \,, \qquad a, b = 1, 2, 3, 4 \,, \tag{2.3}$$

where the metric on the four-dimensional base $\mathcal{H}_B$ is

$$ds^2_{\mathcal{H}_B} = g_{ab} dx^a dx^b \,, \qquad a, b = 1, 2, 3, 4 \,. \tag{2.4}$$

Now, if the four-dimensional space $\mathcal{H}_B$ is a complex Kähler surface, with metric g_{ab}, then the 1-form $A_a dx^a$ is the U(1) connection with field strength F proportional to the Kähler two-form ω of $\mathcal{H}_B$, viz

$$F = i\omega \,. \tag{2.5}$$

With the above ansätz, the condition for the existence of a covariant Killing spinor in the absence of D7–branes is found to be [20]

$$R_{ab} = 6 g_{ab} \,, \tag{2.6}$$

where R_{ab} denotes the Ricci tensor of the metric g_{ab} in (2.4). Thus, the question of finding out admissible background solutions reduces to the question of existence of Kähler–Einstein metric on the base $\mathcal{H}_B$.

One can use the known four-dimensional spaces admitting Kähler–Einstein metric as candidates for $\mathcal{H}_B$ [4]. Some of the choices are related to different partial resolutions of the moduli space of a D3–brane on an orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, studied earlier [15, 14]. The connections are forged by considering the toric data of the D-brane orbifolds of $\mathbb{C}^3$ in the blown-down limit and then realizing the horizon $\mathcal{H}$ as a line-bundle over a certain base $\mathcal{H}_B$ inside the orbifold viewed as the level set of a combination of moment maps obtained by splitting the original one at blown-down [4]. Another interesting case is when the D-brane orbifold is $\mathbb{C}^3/\mathbb{Z}_3$. The corresponding base $\mathcal{H}_B$ is $\mathbb{CP}^2$. In this case (and some others studied in [4]), the horizon $\mathcal{H}$ has the structure of a Sasaki–Einstein manifold, with the aforementioned U(1) providing the corresponding contact structure. Moreover, the action of the U(1) is regular, For such cases the existence of Kähler–Einstein metrics on the base $\mathcal{H}_B$ has been studied earlier [22, 4]. Let us point out that many of the mathematical results used in our considerations are strictly valid for compact spaces. However, we continue to use these results for the non-compact spaces at hand as the compact spaces approximate their non-compact counterparts in a sufficiently small neighborhood of the singular point.

3. Kähler–Einstein metrics on toric varieties

In this section we introduce the canonical toric metric [24] and its deformation [25, 26]. We also derive the equations governing the deformation by demanding the resulting metric to be Einstein. On the other hand, the Kählerity of the deformed metric follows from its very construction.

Let (X, ω) be a compact, connected $2d$-dimensional manifold. Let $\tau : \mathbb{T}^d \longrightarrow \mathrm{Diff}(X, \omega)$ be an effective Hamiltonian action of the standard d-torus $\mathbb{T}^d$. Let $\mu : X \longrightarrow \mathbb{R}^d$ denote the associated moment-map and Δ the image of X on $\mathbb{R}^d$ under the moment map, $\Delta = \mu(X) \subset \mathbb{R}^d$. The convex polytope Δ is referred to as the *moment polytope*. The triple (X, ω, τ) is determined up to isomorphism by the moment polytope [24]. The polytope Δ in $\mathbb{R}^d$ is called *Delzant* if there are d edges meeting at each vertex p of Δ and any edge meeting at p can be given the form $p + sv_i$, for $0 \le s \le \infty$, where $\{v_i\}$ is a basis of $\mathbb{Z}^d$ [24, 25]. Conversely, one can associate a toric variety X_Δ with the above properties to a Delzant polytope Δ in $\mathbb{R}^d$, such that Δ is the moment polytope of X_Δ. Let $X_\Delta = X$ be the toric variety associated to Δ that is, $X_\Delta = \mu^\star(\Delta)$, where $\mu^\star$ denotes the pull-back of μ.

The moment polytope Δ can be described by a set of inequalities of the form $\langle y, u_i \rangle \ge \lambda_i$, $i = 0, 1, 2, \ldots, d-1$. Here u_i denotes the inward-pointing normal to the i-th $(d-1)$-dimensional face of Δ and is a primitive element of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. The pairing $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and y represents a d-dimensional real vector. We can thus define a set of linear maps, $\ell_i : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\ell_i(y) = \langle y, u_i \rangle - \lambda_i, \quad i = 0, 1, \ldots, d-1 \,. \tag{3.1}$$

Denoting the interior of Δ by Δ°, $y \in \Delta^\circ$, if and only if $\ell_i(y) > 0$ for all i.

On the open $\mathbb{T}^d_{\mathbb{C}}$-orbit in X_Δ, associated to a Delzant polytope Δ, the Kähler form ω can be written as [24, 25]:

$$\omega = i\ \partial\bar{\partial}\mu^\star \left(\sum_{i=0}^{d-1} \lambda_i \ln \ell_i + \ell_\infty \right), \tag{3.2}$$

where we have defined ℓ_∞ as the sum,

$$\ell_\infty = \sum_{i=0}^{d-1} \langle y, u_i \rangle \,. \tag{3.3}$$

The potential in the expression (3.2) for the Kähler form ω is determined by the equations for faces of the Delzant polytope (3.1). If ω is a $\mathbb{T}^d$-invariant Kähler form on the complex torus $\mathcal{M} = \mathbb{C}^d/2\pi i\mathbb{Z}^d$, then there exists a function $\mathcal{F}(x)$, with $x = \mathfrak{Re} z$, $z \in \mathbb{C}$, on $\mathcal{M}$, such that $\omega = 2i\partial\bar{\partial}\mathcal{F}$. Moreover, the moment map $\mu : \mathcal{M} \longrightarrow \mathbb{R}^d$ is given by

$$\mu(z) = \frac{\partial \mathcal{F}}{\partial x} \,. \tag{3.4}$$

The Kähler form ω can be written in terms of this new function (Kähler potential) as:

$$\omega = \frac{i}{2} \sum_{j,k=0}^{n-1} \frac{\partial^2 \mathcal{F}}{\partial x_j \partial x_k} dz_j \wedge d\bar{z}_k \,. \tag{3.5}$$

The moment-map μ defines a Legendre transform through (3.4) and thus we can define a new set of variables y_i, $i = 0, \ldots, d$,

$$y_i = \frac{\partial \mathcal{F}}{\partial x_i} \,, \tag{3.6}$$

conjugate to $x_i = \mathfrak{Re}\; z_i$. Consequently, we can define a potential $\mathcal{G}$, conjugate to $\mathcal{F}$ under the Legendre transform (3.6) on Δ°, such that [24, 25],

$$\mathcal{G} = \frac{1}{2} \sum_{k=0}^{d-1} \ell_k(y) \ln \ell_k(y) \,. \tag{3.7}$$

The inverse of the Legendre transform (3.6) can be shown to be of the form [24]:

$$x_i = \frac{\partial \mathcal{G}}{\partial y_i} + r_i \,, \qquad i = 0, \ldots, d-1 \,, \tag{3.8}$$

where r_i are constants. This means that up to a linear term in the coordinates y_i, $\mathcal{G}$ is the Kähler potential Legendre-dual to $\mathcal{F}$. Moreover, the $d \times d$ matrix

$$\mathcal{G}_{ij} = \frac{\partial^2 \mathcal{G}}{\partial y_i \partial y_j} \,, \tag{3.9}$$

evaluated at $y_i = \partial \mathcal{F}/\partial x_i$ (3.6), is the inverse of the matrix

$$\mathcal{F}_{ij} = \frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j} \,. \tag{3.10}$$

The Ricci-tensor for the metric (3.10) takes the following form:

$$R_{ij} = -\frac{1}{2}\frac{\partial^2 \ln\det\ \mathcal{F}}{\partial x_i \partial x_j} \tag{3.11}$$

$$= -\frac{1}{2}\sum_{k,l=0}^{n-1} \mathcal{G}^{lj}\frac{\partial^2 \mathcal{G}^{ik}}{\partial y_k \partial y_l}\,, \tag{3.12}$$

where $\mathcal{G}^{ij}$ denotes the inverse of $\mathcal{G}_{ij}$. Note that, $\mathcal{F}$ in (3.11) (and in (3.13) below) denotes the matrix $\mathcal{F}_{ij}$, and not the Kähler potential unlike elsewhere in this note. The Ricci-scalar for this metric is then derived by multiplying (3.12) with $\mathcal{G}_{ij}$, which is the inverse of the metric $\mathcal{F}_{ij}$ in the y coordinates, and is given by [25]

$$R = -\frac{1}{2}\sum_{i,j=0}^{n-1} \mathcal{F}^{ij}\frac{\partial^2 \ln\det \mathcal{F}}{\partial x_i \partial x_j} \tag{3.13}$$

$$= -\frac{1}{2}\sum_{i,j=0}^{n-1} \frac{\partial^2 \mathcal{G}^{ij}}{\partial y_i \partial y_j}\,, \tag{3.14}$$

where $\mathcal{F}^{ij}$ denotes the inverse of $\mathcal{F}_{ij}$. For our purposes, it will be convenient to use the matrix $\mathcal{G}_{ij}$ in the coordinates y. One can, in principle, rewrite all the relevant expressions in terms of the coordinates x_i and the matrix $\mathcal{F}_{ij}$.

The expressions for the Kähler potential (3.2), the metric (3.10) and the curvature (3.12) described above will be referred to as the *canonical* ones. Moreover, if $\iota : X \longrightarrow \mathbb{CP}^n$, denotes a projective embedding of X in $\mathbb{CP}^n$ for some n, then the canonical Kähler form ω, given by (3.2), is $\mathbb{T}^d$-equivariantly symplectomorphic to the pull-back of the Kähler form ω_{FS} corresponding to the Fubini-Study metric on $\mathbb{CP}^n$, namely $\iota^\star\omega_{\text{FS}}$.

However, the curvature of the canonical toric metric is arbitrary in general. In order to obtain a metric with prescribed — usually constant — curvature, as in our cases, one may need to deform the canonical Kähler form to another one in the same Kähler class. This may be effected by adding a function, smooth on some open subset of $\mathbb{R}^n$ containing Δ, to the potential $\mathcal{G}$, such that the Hessian of the new potential is positive definite on Δ° in order for the new potential to be Kähler. This furnishes a deformed Kähler metric in the same Kähler class as the canonical one. The variety X_Δ is then endowed with two different Kähler forms related by a $\mathbb{T}^n$-equivariant symplectomorphism since the function f is non-singular. One then finds out the form of this extra function by demanding the prescribed curvature. This method was used in deriving the extremal Kähler Metric on $\mathbb{CP}^2\#\mathbb{CP}^2$ in Calabi's form [25] as well as in finding out a Ricci-flat metric on the resolved D–brane orbifold $\mathbb{C}^3/\mathbb{Z}_3$ [26]. See [27] for another approach involving the Heat equation. Following the approach of [25, 26], let us as add a function f to $\mathcal{G}$, and define a new potential

$\widetilde{\mathcal{G}} = \mathcal{G} + \frac{1}{2} f$. Now the matrix $\widetilde{\mathcal{G}}_{ij}$ corresponding to the potential $\widetilde{\mathcal{G}}$ assumes the form

$$\widetilde{\mathcal{G}}_{ij} = \mathcal{G}_{ij} + \frac{1}{2} \frac{\partial^2 f}{\partial y_i \partial y_j} \,. \tag{3.15}$$

One can then find out the Kähler metric by inverting $\widetilde{\mathcal{G}}_{ij}$ and hence the curvature for this new metric. This will also give rise to a new $\widetilde{\mathcal{F}}$ corresponding to $\mathcal{F}$ and also new coordinates $\widetilde{x}$. What we propose to do next is to write down the general form of the metric for a function f and then determine f by demanding that the Ricci-tensor given by the formula (3.12) to be proportional to the new metric $\widetilde{\mathcal{G}}^{ij}$. This yields a differential equation for the function f.

For the two cases considered here, the base $\mathcal{H}_B$ is two-(complex)-dimensional. Thus, we have $i = 1, 2$. Specialising to this case, the three components of R_{ij} can be written as:

$$-2R_{11} = \widetilde{\mathcal{G}}^{11} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_1} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_2} \right) + \widetilde{\mathcal{G}}^{12} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_2 \partial y_2} \right), \tag{3.16}$$

$$-2R_{12} = \widetilde{\mathcal{G}}^{12} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_1} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_2} \right) + \widetilde{\mathcal{G}}^{22} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_2 \partial y_2} \right), \tag{3.17}$$

$$-2R_{22} = \widetilde{\mathcal{G}}^{22} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{22}}{\partial y_2 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_2} \right) + \widetilde{\mathcal{G}}^{12} \left(\frac{\partial^2 \widetilde{\mathcal{G}}^{22}}{\partial y_1 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_1} \right). \tag{3.18}$$

If there exists a Kähler–Einstein metric on $\mathcal{H}_B$, then for some choice of the deforming function the components of the curvature R_{ij} must be equal to the components of the metric $\widetilde{\mathcal{G}}^{ij}$. We find out a solution for f by imposing this condition on R_{ij}. Thus, the condition for the manifold $\mathcal{H}_B$ to be Kähler–Einstein, i.e.

$$R_{ij} = \Lambda \widetilde{\mathcal{G}}^{ij} \,, \tag{3.19}$$

is satisfied if

$$\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_1} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_2} = -2\Lambda \,, \tag{3.20}$$

$$\frac{\partial^2 \widetilde{\mathcal{G}}^{22}}{\partial y_2 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_2} = -2\Lambda \,, \tag{3.21}$$

$$\frac{\partial^2 \widetilde{\mathcal{G}}^{22}}{\partial y_1 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_1 \partial y_1} = 0 \,, \tag{3.22}$$

$$\frac{\partial^2 \widetilde{\mathcal{G}}^{11}}{\partial y_1 \partial y_2} + \frac{\partial^2 \widetilde{\mathcal{G}}^{12}}{\partial y_2 \partial y_2} = 0 \,. \tag{3.23}$$

Let us note that the non-covariant notation for indices in (3.19) originates from the fact that, abiding by common practice, we have denoted derivatives by subscripts and the metric under consideration has been written as $\widetilde{\mathcal{G}}^{ij} = \widetilde{\mathcal{G}}_{ij}^{-1}$ in y variables. Here Λ is a constant parameter that determines the Ricci scalar as $R = 2\Lambda$. However, in the following we keep this parameter arbitrary for book-keeping. In sections 4 and 5, we use equations (3.20)–(3.23) to prove the existence of a Kähler–Einstein metric on $\mathcal{H}_B$ for the two cases mentioned earlier.

4. Kähler–Einstein metric on the base $\mathcal{H}_B$ of $\mathbb{C}^3/\mathbb{Z}_3$

In this section we consider the blown-down limit of the moduli space of a D3–brane on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. We first find out a toric description of the base $\mathcal{H}_B$ staring from the toric data of $\mathbb{C}^3/\mathbb{Z}_3$. Then, following the discussion in section 3, we find out the Kähler–Einstein metric on this base.

In order to obtain the base $\mathcal{H}_B$ for blown-down $\mathbb{C}^3/\mathbb{Z}_3$, let us start from the charges of the fields. The toric data derived from F- and D-flatness conditions of the gauge theory is given by [12]

$$\mathcal{T} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} . \tag{4.1}$$

The blown-down moduli space is obtained in terms of charges specified by the kernel of $\mathcal{T}$ with the resolution (Fayet–Iliopoulos) parameter vanishing. The charge matrix becomes

$$\mathcal{Q} = (\mathrm{Ker}\ \mathcal{T})^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & -3 \end{pmatrix} , \tag{4.2}$$

where a superscript T designates matrix transpose. The resulting moduli space is described by the following moment-map equation:

$$|z_0|^2 + |z_1|^2 + |z_2|^2 - 3|z_3|^2 = 0 , \tag{4.3}$$

where $\{z_i | i = 0, 1, 2, 3\}$ are the homogeneous variables on the corresponding toric variety. The horizon $\mathcal{H}$ is obtained as a line-bundle from (4.3), following the construction of [4]

$$|z_0|^2 + |z_1|^2 + |z_2|^2 = \zeta \qquad \text{and} \qquad 3|z_3|^2 = \zeta . \tag{4.4}$$

The parameter ζ is left arbitrary in (4.4) for book-keeping. It can be set to unity. Equations (4.4) describe a space $\mathbb{S}^5 \times \mathbb{S}^1$. The horizon $\mathcal{H}$ is obtained from (4.4) after quotienting by a U(1) as $(\mathbb{S}^5 \times \mathbb{S}^1)/\mathrm{U}(1)$, with the U(1)-action on the homogeneous variables given by

$$\mathrm{U}(1) : (z_0, z_1, z_2, z_3) \longmapsto (e^{i\theta} z_0, e^{i\theta} z_1, e^{i\theta} z_2, e^{-3i\theta} z_3) . \tag{4.5}$$

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Now, to find the toric description for the base $\mathcal{H}_B$, we collect the charges from (4.4) into another charge-matrix

$$\mathcal{Q}_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{4.6}$$

and find out the toric data corresponding to $\mathcal{H}_B$ as the co-kernel of the transpose of $\mathcal{Q}_B$. The resulting toric data is given as

$$\mathcal{T}' = \text{coKer } \mathcal{Q}_B^{\mathrm{T}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \tag{4.7}$$

Leaving out the last column, this can be recognised as the toric data for $\mathbb{CP}^2$, namely,

$$\mathcal{T}_B = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{4.8}$$

This description corresponds to the "most efficient" description of the toric variety $\mathcal{H}_B$, as mentioned in [4]. The two-vectors described by the columns of $\mathcal{T}'$ are plotted in figure 1. As in [4], the toric data $\mathcal{T}_B$ in (4.8) for $\mathcal{H}_B$ is obtained after omitting the point in the interior of the triangle. Moreover, the toric data $\mathcal{T}'$ in (4.7) can be identified as the first two rows in the toric data $\mathcal{T}$ in (4.1) for $\mathbb{C}^3/\mathbb{Z}_3$. We shall see shortly that the omission of the point in the interior of the polygon is related to the construction of the Delzant polytope corresponding to the variety.

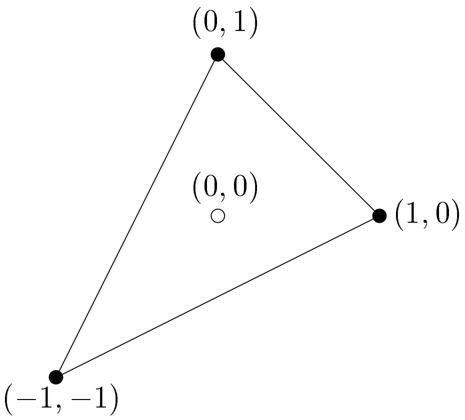

Figure 1: Plot of the columns of the toric data $\mathcal{T}'$ for $\mathbb{C}^3/\mathbb{Z}_3$: the unfilled circle is omitted from the description in obtaining the Delzant polytope. The filled ones correspond to $\mathcal{T}_B$.

The corresponding canonical toric metric from $\mathcal{T}_B$ in (4.8) is the Fubini-Study metric, as given in [25]. The Delzant polytope is given as

$$y_1 \geq 0, \qquad y_2 \geq 0\,, \qquad \zeta - y_1 - y_2 \geq 0\,, \tag{4.9}$$

where we have shifted the variables y_i, $i = 1, 2$ by the respective values of the support functions a_i, $i = 1, 2$, at each of the one-dimensional cone generators specified by the columns of $\mathcal{T}_B$, following [26]. The Kähler structure ζ is given as $\zeta = a_0 + a_1 + a_2$, and hence the first Chern-class of the variety, calculated as the sum of the coefficients of a_i in ζ [24], is constant, $\mathfrak{c}_1(X_\Delta) = 1+1+1 = 3$. Thus, the variety in question is Fano. This corroborates to the fact that the base $\mathcal{H}_B$ is $\mathbb{CP}^2$, as pointed out in [4]. Let us point out that writing down the Delzant polytope provides a rationale for the omission of the interior points of the polygon shown in figure 1 (this, however, is *not* the Delzant polytope): they do not yield any new face for the Delzant polytope. The last column of (4.7) does not affect the polytope (4.9). The canonical potential $\mathcal{G}$ corresponding to the Delzant polytope (4.9) is, by (3.7),

$$\mathcal{G} = \frac{1}{2}\Big[y_1 \ln y_1 + y_2 \ln y_2 + (\zeta - y_1 - y_2)\ln(\zeta - y_1 - y_2)\Big] . \tag{4.10}$$

The canonical metric ensuing from the potential (4.10) is given by

$$\mathcal{G}^{ij} = \frac{2}{\zeta}\begin{pmatrix} y_1(\zeta - y_1) & -y_1 y_2 \\ -y_1 y_2 & y_2(\zeta - y_2) \end{pmatrix} . \tag{4.11}$$

The Ricci-tensor evaluated from (4.11), using (3.12) can be checked to be proportional to the metric (4.11). That is, the equations (3.20)–(3.23) are satisfied with $f = 0$ and $\Lambda = 3/\zeta$. The Ricci-scalar for this metric is found from this as

$$R = -\frac{1}{2}\frac{2}{\zeta}(-2 - 1 - 1 - 2) = \frac{6}{\zeta} = 2\Lambda , \tag{4.12}$$

as expected from (3.14). Thus, we have verified that there does exist a Kähler–Einstein metric on the base $\mathcal{H}_B$ of the blown-down D–brane orbifold $\mathbb{C}^3/\mathbb{Z}_3$. The metric is diffeomorphic to the Fubini-Study metric, with a constant scalar curvature as given in (4.12).

A comment is in order. A Ricci-flat metric on the resolved D3–brane orbifold $\mathbb{C}^3/\mathbb{Z}_3$ was found out earlier using similar techniques in [26]. This needed a deformation of the canonical toric metric by a function which was shown to be a solution of the differential equation[1]

$$f'' = \frac{9y_3(\xi + 3y_3)^3 - (\xi + 12y_3)[c + (\xi + 3y_3)^3]}{y_3(\xi + 3y_3)[c + (\xi + 3y_3)^3]} , \tag{4.13}$$

where y_3 denotes the third variable needed to form the Delzant polytope of the three-dimensional variety, c is a constant and ξ is the resolution parameter (set to zero in (4.3)). The metric on the complex cone $C(\mathcal{H})$ constructed from the metric (4.11) using (2.2) does not appear to be in the same form as the one derived in [26] for the resolved D–brane orbifold $\mathbb{C}^3/\mathbb{Z}_3$, with vanishing Kähler class. This discrepancy, however, may be attributed to the fact that constancy of the scalar curvature on the base was not imposed on the metric in [26], as was pointed out there. However, since the restrictions of the metric (2.2), and the one derived in [26] on the base $\mathcal{H}_B$ are the same, with suitable normalisation of y_3, there exists some neighborhood of $\mathcal{H}_B$ in $\mathbb{C}^3/\mathbb{Z}_3$ on which the two metrics are diffeomorphic to one another, thanks to the Darboux-Weinstein theorem [28].

[1]The expression (4.13) corrects a typographical error in equation (61) in [26]

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5. Kähler–Einstein metric on the base $\mathcal{H}_B$ of $\mathbb{C}^3/\mathbb{Z}_5$

In this section we consider the blown-down limit of the moduli space of a D3–brane on $\mathbb{C}^3/\mathbb{Z}_5$. First, let us find out the toric description of the base $\mathcal{H}_B$, from the toric data of the moduli space [12].

We shall mimic the considerations of section 4. Let us start from the following toric data for the orbifold $\mathbb{C}^3/\mathbb{Z}_5$ [12],

$$\mathcal{T} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & -1 \\ 5 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{5.1}$$

The corresponding charge matrix, evaluated as the transpose of the kernel of $\mathcal{T}$ takes the form

$$\begin{pmatrix} 1 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}, \tag{5.2}$$

that leads to two moment map equations in the blown-down limit with vanishing Fayet–Iliopoulos parameters of the gauge theory:

$$|z_0|^2 + |z_1|^2 + |z_3|^2 - 3|z_4|^2 = 0 \tag{5.3}$$

$$|z_2|^2 - 2|z_3|^2 + |z_4|^2 = 0\,. \tag{5.4}$$

From the charge matrix (5.2), one obtains by row operations (changing basis) the charge matrix

$$\mathcal{Q} = \begin{pmatrix} 1 & 1 & 3 & -5 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}. \tag{5.5}$$

$\mathcal{Q}$ corresponds to eliminating z_4 from (5.3) by using (5.4) and leads to the following equation for the global moment map:

$$|z_0|^2 + |z_1|^2 + 3|z_2|^2 - 5|z_3|^2 = 0\,. \tag{5.6}$$

The horizon $\mathcal{H}$ is obtained from (5.6), or from the first row of $\mathcal{Q}$ constructing the line-bundle [4]:

$$|z_0|^2 + |z_1|^2 + 3|z_2|^2 = \zeta \qquad \text{and} \qquad 5|z_3|^2 = \zeta\,. \tag{5.7}$$

Again, we have kept a free-parameter ζ for book-keeping. Equations (5.7) describe a direct product $\mathbb{S}^5 \times \mathbb{S}^1$. The corresponding horizon is obtained from (5.7) quotienting by a U(1) as $\mathcal{H} = (\mathbb{S}^5 \times \mathbb{S}^1)/\mathrm{U}(1)$, with the U(1)-action given by

$$\mathrm{U}(1):\ (z_0, z_1, z_2, z_3) \longmapsto (e^{i\theta} z_0, e^{i\theta} z_1, e^{3i\theta} z_2, e^{-5i\theta} z_3)\,. \tag{5.8}$$

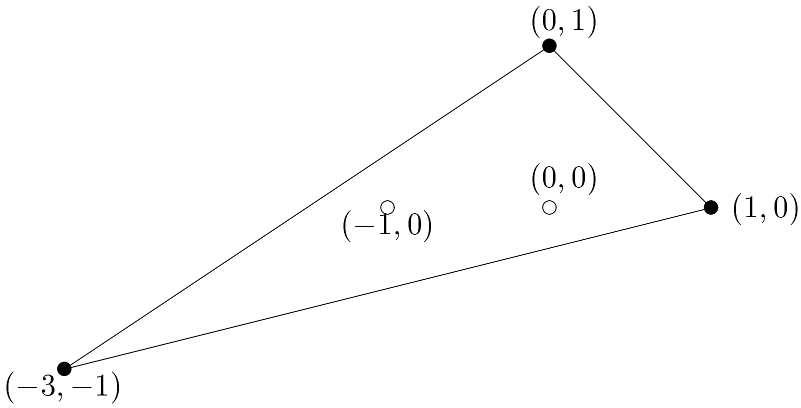

Figure 2: Plot of the columns of the toric data $\mathcal{T}'$ for $\mathbb{C}^3/\mathbb{Z}_5$: the unfilled circles are omitted from the description in obtaining the Delzant polytope. The filled ones correspond to $\mathcal{T}_B$.

This U(1)-action (5.8) is not regular. For instance, the orbits of the points $(0,0,1,0)$ and $(1,0,0,0)$ are of different lengths. However, we proceed as before to derive the base $\mathcal{H}_B$ from this, and to this end write the corresponding charge-matrix as

$$\mathcal{Q}_B = \begin{pmatrix} 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} . \tag{5.9}$$

This leads to the following toric data obtained as the co-kernel of the transpose of $\mathcal{Q}_B$, namely

$$\mathcal{T}' = \begin{pmatrix} 0 & -3 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} . \tag{5.10}$$

From $\mathcal{T}'$, ignoring the last two columns, corresponding to points in the interior of the triangle, as shown in figure 2, we obtain the toric data for $\mathcal{H}_B$ as the following rectangular matrix:

$$\mathcal{T}_B = \begin{pmatrix} 0 & -3 & 1 \\ 1 & -1 & 0 \end{pmatrix} . \tag{5.11}$$

As before, one can identify the matrix $\mathcal{T}'$ in (5.10) inside $\mathcal{T}$ in (5.1): $\mathcal{T}'$ is $\mathcal{T}$ with the last row omitted.

Having obtained the toric description for the base $\mathcal{H}_B$, we now proceed as in section 4, to find out the toric metric. We first derive the canonical toric metric and the scalar curvature to learn that the latter is not constant, signalling that the canonical toric metric is not Kähler–Einstein. Following the considerations of section 3, we then deform the canonical toric metric and determine the deformation imposing the condition that the deformed metric is Kähler–Einstein, as discussed in section 3. Let us start by considering the Delzant polytope corresponding to the variety $\mathcal{H}_B$ as expressed in terms of the toric data $\mathcal{T}_B$ in (5.11). The Delzant polytope Δ is defined by the following inequalities dictated by the columns of $\mathcal{T}_B$:

$$y_1 \geq 0\,, \qquad y_2 \geq 0\,, \qquad \zeta - 3y_1 - y_2 \geq 0\,. \tag{5.12}$$

Again the variables are shifted by the values of the respective support functions at the one-dimensional cone generators, given by the columns on $\mathcal{T}_B$. This leads to $\zeta = a_0 + 3a_1 + a_2$, implying that the first Chern-class $\mathfrak{c}_1(X_\Delta) = 1 + 3 + 1 = 5$, whence the variety $\mathcal{H}_B$ is found to be Fano. Once again the omission of the last two columns of $\mathcal{T}'$ is justified by their being inconsequential for constructing the Delzant polytope.

The canonical toric metric can be obtained from the potential

$$\mathcal{G} = \frac{1}{2}\Big[y_1 \ln y_1 + y_2 \ln y_2 + (\zeta - 3y_1 - y_2)\ln(\zeta - 3y_1 - y_2)\Big]\,, \tag{5.13}$$

and the corresponding matrix $\mathcal{G}_{ij}$ is

$$\mathcal{G}_{ij} = \frac{1}{2A}\begin{pmatrix} 9 + \frac{A}{y_1} & 3 \\ 3 & 1 + \frac{A}{y_2} \end{pmatrix}\,, \tag{5.14}$$

where we have defined $A = \zeta - 3y_1 - y_2$. The metric $\mathcal{G}^{ij}$ derived by inverting (5.14) is not Einstein. Indeed, the Ricci scalar for this canonical toric metric is:

$$R = \frac{26\zeta^2 + 60\zeta y_1 + 72y_1^2}{(\zeta + 6y_1)^3}\,. \tag{5.15}$$

The Ricci scalar R is, however, positive definite, since, by (5.12), $y_1 \geq 0$ on the Delzant polytope Δ.

Thus, as mentioned in section 3, let us now look for a deformed Kähler metric which is Einstein, by deforming the above potential $\mathcal{G}$ by a function f. As a simplifying ansätz, we choose f to be a function of y_1 only, $f = f(y_1)$, since the Ricci-scalar (5.15) depends only on y_1. This leads to

$$\widetilde{\mathcal{G}}_{ij} = \frac{1}{2A}\begin{pmatrix} 9 + \frac{A}{y_1} + Af'' & 3 \\ 3 & 1 + \frac{A}{y_2} \end{pmatrix}\,, \tag{5.16}$$

where a prime signifies differentiation with respect to y_1. Inverting $\widetilde{\mathcal{G}}_{ij}$, we obtain the following expression for $\widetilde{\mathcal{G}}^{ij}$:

$$\widetilde{\mathcal{G}}^{ij} = \begin{pmatrix} \frac{2y_1}{F}(\zeta - 3y_1) & -\frac{6y_1y_2}{F} \\ -\frac{6y_1y_2}{F} & 2y_2(1 - \frac{y_2\phi}{F}) \end{pmatrix}\,, \tag{5.17}$$

where $F = \zeta + 6y_1 + y_1 f''(\zeta - 3y_1)$ is a nowhere-vanishing function and we have defined $\phi = (F - 9y_1)/(\zeta - 3y_1)$. Now we solve (3.20)–(3.23) using the expression (5.17) for $\widetilde{\mathcal{G}}^{ij}$. Equation (3.23) is identically satisfied as f does not depend on y_2. Equations (3.21) and (3.22) lead to two equations respectively,

$$-4\left(\frac{\phi}{F}\right) - 6\left(\frac{y_1}{F}\right)' = -2\Lambda, \tag{5.18}$$

$$-4\left(\frac{\phi}{F}\right)' - 6\left(\frac{y_1}{F}\right)'' = 0\,. \tag{5.19}$$

In deriving (5.19) from (3.22), we have used the fact that $y_2 \neq 0$, i.e. $y \in \Delta^\circ$. Equation (5.19) can be obtained by differentiating (5.18) with respect to y_1. Finally, from (3.20) one derives,

$$(\zeta - 3y_1)\left(2y_1 F'^2 - y_1 F F''\right) + (15y_1 - 2\zeta)FF' - 9F^2 + \Lambda F^3 = 0\,, \tag{5.20}$$

that, however, is solved identically on using (5.18) and (5.19). Thus, one is left with one single equation for f, namely (5.18), that can be rewritten as

$$3y_1 F' - (2\phi + 3)F + \Lambda F^2 = 0\,. \tag{5.21}$$

Using the expression for ϕ, equation (5.21) takes the following form:

$$3y_1(\zeta - 3y_1)F' - 3(\zeta - 9y_1)F + \Big[(\zeta - 3y_1)\Lambda - 2\Big]F^2 = 0\,. \tag{5.22}$$

In order to solve this, we rewrite it further in terms of a new function $\chi = 1/F$ as:

$$3y_1(\zeta - 3y_1)\chi' + 3(\zeta - 9y_1)\chi - \Big[(\zeta - 3y_1)\Lambda - 2\Big] = 0\,. \tag{5.23}$$

Equation (5.23) is a linear first-order differential equation in y_1 and can be solved for χ and thence for F resulting in

$$F = \frac{9y_1}{\Theta}\,, \tag{5.24}$$

where $\Theta = 1 - \frac{\Lambda}{3}(\zeta - 3y_1) + 9c(\zeta - 3y_1)^{-2}$ and c is a constant of integration. One is then lead to the following equation for f:

$$f'' = \frac{9y_1 - (\zeta + 6y_1)\Theta}{y_1(\zeta - 3y_1)\Theta} \tag{5.25}$$

that can be explicitly integrated to determine the potential $\widetilde{\mathcal{G}}$ in y-variables. Using (5.25) in the expression (5.17) for $\widetilde{\mathcal{G}}^{ij}$ yields the desired Kähler–Einstein metric on the base $\mathcal{H}_B$ of the cone $\mathbb{C}^3/\mathbb{Z}_5$.

6. Conclusion

To conclude, in this note we have demonstrated the existence of Kähler–Einstein metrics on the bases of the complex cones obtained as moduli spaces of D3–branes on $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$ orbifolds. These cones are, in turn, metric cones over horizons $\mathcal{H}$, and can be used as the internal space in Type–IIB string theory on AdS^5, in the context of the conjectured AdS-CFT correspondence. The analysis validates the candidature of the horizon obtained from the blown-down orbifold $\mathbb{C}^3/\mathbb{Z}_5$ along with $\mathbb{C}^3/\mathbb{Z}_3$, the latter having already been pointed out in [4]. It seems possible to generalize the considerations in this note to establish similar results for the existence of Kähler–Einstein metrics on $\mathcal{H}_B$ for other D3–brane orbifolds already studied in literature [13, 14, 15, 4]. However, it is not clear how to extend this analysis to cases that do not admit a toric description. The metrics presented here, like the one in [26], are in rather special *real* variables. It will be interesting to be able to write these metrics in terms of variables better-suited to studying their geometric properties. We hope to return to this issue in future.

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